J. Fluid Mech. (1967), *vol.* 29, *part* 1, *pp.* 1–16
Printed in Great Britain

Linear theory of rotating stratified fluid motions

By V. BARCILON AND J. PEDLOSKY
Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received 9 June 1966)

A linear theory for steady motions in a rotating stratified fluid is presented, valid under the assumption that $\epsilon < E$, where ϵ and E are respectively the Rossby and Ekman numbers. The fact that the stable stratification inhibits vertical motions has important consequences and many features of the dynamics of homogeneous rotating fluids are no longer present. For instance, in addition to the absence of the Taylor–Proudman constraint, it is found that Ekman layer suction no longer controls the interior dynamics. In fact, the Ekman layers themselves are frequently absent. Furthermore, the vertical Stewartson boundary layers are replaced by a new kind of boundary layer whose structure is characteristic of rotating stratified fluids. The *interior* dynamics are found to be controlled by dissipative processes.

1. Introduction

The following three features, which give the theory of homogeneous rotating fluids its distinctive character, are found to be essential for an understanding of a wide variety of phenomena.

The first feature is the Taylor–Proudman (1917) theorem, which is valid for a homogeneous fluid in which the viscous and inertial forces are small compared to the Coriolis force. It states that the velocity does not vary in the direction of $\mathbf{\Omega}$, where $\mathbf{\Omega}$ is the rotation vector. It also implies that very slight extensions of fluid columns parallel to $\mathbf{\Omega}$ can induce appreciable vorticity throughout the fluid.

The second important feature is that the extension of fluid columns in the fluid's inviscid interior can be produced by the suction of fluid into thin viscous boundary layers (Ekman layers) existing along surfaces which are not parallel to $\mathbf{\Omega}$. In fact, this boundary-layer suction provides a dynamical mechanism whereby the viscous boundary layers can transmit boundary information into the fluid by vortex tube stretching. As a result, the Ekman layers can strongly control the inviscid, interior motion, since their effects are inertial rather than diffusive in character. The spin-up problem discussed by Greenspan & Howard (1963) and Greenspan (1964, 1965) illustrates dramatically this boundary-layer control.

Thirdly, the circulations produced by the Ekman layer suction must frequently be closed by boundary layers parallel to $\mathbf{\Omega}$ which were first discussed by Stewartson (1957). These boundary layers have a double structure and can either be free, or attached to a rigid wall which is parallel to the rotation vector.

In the present paper we propose to examine how these features are modified by stratification. For the sake of simplicity, we shall restrict our attention to steady motions and to the case in which $\mathbf{\Omega}$ and $\mathbf{g}$ are aligned, where $\mathbf{g}$ is the gravitational acceleration. Furthermore, we shall only consider cylindrical boundaries whose horizontal cross-sections are either a circle, or an annular region bounded by two concentric circles. However, most of our results will be valid for more general container configurations.

The plan to be followed is this: we shall first formulate (§ 2) a general class of boundary-value problems for either thermally or mechanically driven flows which can be regarded as small perturbations away from a state of rigid rotation and linear vertical stratification. Making a perturbation expansion in the Ekman number, we shall then examine the fluid's interior (§ 3) in order to derive interior equations. From an investigation of the horizontal and vertical boundary layers (§§ 4 and 5) for a variety of boundary conditions, we shall then deduce the boundary conditions which the *interior fields* must satisfy. We shall finally be able to formulate simpler boundary-value problems for the interior fields (§ 6).

In addition to the absence of the Taylor–Proudman constraint, we shall show that under certain conditions the mechanism of vortex tube stretching by Ekman layer suction is no longer allowed. In fact, the Ekman layers will either turn out to be absent to lowest order, or to play a purely passive role in the dynamics. Furthermore, a new kind of vertical boundary layer appears, replacing the Stewartson boundary layers observed in homogeneous fluid. All these drastic changes in behaviour of the fluid motion are intimately connected with the ability of the imposed stable stratification to inhibit vertical motions.

2. Formulation

The equations governing the steady motions of an incompressible, viscous, heat-conducting fluid, written in a co-ordinate frame rotating with angular velocity Ω about the vertical are:

$$\mathbf{q}.\nabla\mathbf{q}+2\Omega\hat{k}\times\mathbf{q} = -\frac{1}{\rho}\nabla p - g\hat{k} + \nabla\left(\frac{\Omega^2}{2}|\hat{k}\times\mathbf{r}|^2\right) + \nu\nabla^2\mathbf{q},$$

$$\nabla.\mathbf{q} = 0,$$

$$\mathbf{q}.\nabla T = \kappa\nabla^2 T,$$

$$\rho = \rho_0[1-\alpha(T-T_0)].$$

The notations are standard, viz. $\mathbf{q}$, p, ρ and T are respectively the velocity, pressure, density and temperature of the fluid at a point $\mathbf{r}$; ν and κ are the (constant) kinematic viscosity and thermal conductivity, $\hat{k}$ is a unit vertical vector. We have explicitly assumed that it is permissible to use a linear $\rho-T$ relationship as the equation of state, in which α is the coefficient of thermal expansion and ρ_0 and T_0 constant reference values of density and temperature.†

† In this study thermal effects are considered to be the stratification agency. Only minor notational changes are required if some other property, say salinity, effects the stratification.

In the absence of motions relative to the rotating frame, the conditions of static balance imply that:

$$\rho = \rho_s \equiv \rho_{00}\left\{1+\beta\left(z-\frac{\Omega^2}{2g}|\hat{k}\times\mathbf{r}|^2\right)\right\},$$

where β is an arbitrary function. In the present study we shall restrict our attention to those cases for which the parameter Ω^2L/g is sufficiently small so that β may be considered as a function of z only. Moreover, if we assume that β is a linear function of z, then the static temperature is also a solution of the heat equation.

Let us introduce the following dimensionless variables, denoted by asterisks:

$$\mathbf{q} = \epsilon\Omega L\mathbf{q}_*, \quad p = -g\int^z \rho_s(z')\,dz' + \epsilon\Omega^2L^2\rho_s p_*,$$

$$\mathbf{r} = L\mathbf{r}_*, \quad T = T_s(z)+\epsilon\frac{\Omega^2 L}{\alpha g}T_*,$$

where ϵ, which is a measure of the departure from a state of rigid rotation and linear vertical stratification, is the Rossby number. Assuming that ϵ, Ω^2L/g and β are all less than $O(E)$, where $E = \nu/\Omega L^2$ is the Ekman number, and dropping the asterisks, the dimensionless equation of motion correct to $O(E)$ can be written thus:

$$2\hat{k}\times\mathbf{q} = -\nabla p+T\hat{k}+E\nabla^2\mathbf{q}, \tag{2.1}$$

$$\nabla.\mathbf{q} = 0, \tag{2.2}$$

$$\sigma S\hat{k}.\mathbf{q} = E\nabla^2 T. \tag{2.3}$$

The parameter $S = -g\beta_z/\Omega^2L$ is a measure of the stratification and is assumed to be positive and of order one. The Prandtl number $\sigma = \nu/\kappa$ is also considered to be of $O(1)$.

The fluid is contained within a cylindrical region bounded by two horizontal planes at $z = 0$ and $z = 1$, and either one or two vertical cylinders of radius $r = r_1$ and $r = r_2$ such that $r_1 > r_2 \geqslant 0$. On the horizontal boundaries, the velocity of the fluid must equal the prescribed boundary velocities, i.e.

$$\mathbf{q} = \mathbf{q}_B(r,\theta) \quad \text{on} \quad z=0, \qquad \mathbf{q} = \mathbf{q}_T(r,\theta) \quad \text{on} \quad z=1. \tag{2.4}$$

Similarly, on the vertical boundaries:

$$\mathbf{q} = \mathbf{q}_o(z,\theta) \quad \text{at} \quad r=r_1, \qquad \mathbf{q} = \mathbf{q}_I(z,\theta) \quad \text{at} \quad r=r_2. \tag{2.5}$$

Finally, either the temperature, or the heat flux, or a combination of both is also specified, viz.

$$a_B T+b_B T_z = \Theta_B(r,\theta) \quad \text{on} \quad z=0, \tag{2.6a}$$

$$a_T T+b_T T_z = \Theta_T(r,\theta) \quad \text{on} \quad z=1, \tag{2.6b}$$

$$a_o T+b_o T_r = \Theta_o(\theta,z) \quad \text{on} \quad r=r_1, \tag{2.6c}$$

$$a_I T+b_I T_r = \Theta_I(\theta,z) \quad \text{on} \quad r=r_2, \tag{2.6d}$$

where (r,θ,z) are the usual cylindrical co-ordinates.

It is important to note that unless $r_2 = 0$, the fluid region is not simply-connected. Of course when $r_2 = 0$, all the boundary conditions at the 'inner' vertical wall should be disregarded. (2.1)–(2.6) constitute a well-posed boundary-value problem for a large class of either thermally or mechanically driven flows, which we propose to consider for the case of small Ekman numbers.

3. The interior equations

In order to find the equations of motion valid in the fluid's interior, we expand the dependent variables in an asymptotic series in powers of $E^{\frac{1}{2}}$, e.g.

$$\mathbf{q} = \mathbf{q}^{(0)} + E^{\frac{1}{2}}\mathbf{q}^{(1)} + E\mathbf{q}^{(2)} + \dots \qquad (3.1)$$

Note that the assumed series for the interior variables proceeds in powers of $E^{\frac{1}{2}}$ even though the differential equations contain only E. Nevertheless, this form of the expansion is assumed because experience indicates the possibility of boundary-layer corrections to the interior flow which are of $O(E^{\frac{1}{2}})$ (Greenspan 1965).

Substitution of this expansion in (2.1)–(2.3) yields for the $O(1)$ equations:

$$2\hat{k}\times\mathbf{q}^{(0)} = -\nabla p^{(0)}, \qquad (3.2)$$

$$T^{(0)} = \hat{k}\,.\,\nabla p^{(0)}, \qquad (3.3)$$

$$\hat{k}\,.\,q^{(0)} = 0, \qquad (3.4)$$

$$\nabla\,.\,\mathbf{q}^{(0)} = 0. \qquad (3.5)$$

To $O(1)$, the flow is geostrophic and hydrostatic. The vertical velocity is so suppressed by the stratification that an examination of (2.3) shows that it must be of $O(E)$. *The interior motion is therefore constrained by the stratification to be purely horizontal to* $O(E)$. This constitutes a crucial difference between the stratified and homogeneous cases, since in the latter case the interior vertical velocity is usually of $O(E^{\frac{1}{2}})$ rather than of $O(E)$. As we shall presently see, this fact will have important consequences for the Ekman boundary layers.

The $O(E^{\frac{1}{2}})$ equations are:

$$\left.\begin{aligned} 2\hat{k}\times\mathbf{q}^{(1)} &= -\nabla p^{(1)},\\ T^{(1)} &= \hat{k}\,.\,\nabla p^{(1)},\\ \hat{k}\,.\,\mathbf{q}^{(1)} &= 0,\\ \nabla\,.\,\mathbf{q}^{(1)} &= 0. \end{aligned}\right\} \qquad (3.6)$$

Therefore the interior $O(1)$ and $O(E^{\frac{1}{2}})$ dynamics are identical. Elimination of the pressure between (3.2) and (3.3) yields the familiar 'thermal wind' relation, viz.

$$2(\hat{k}\,.\,\nabla)\,\mathbf{q}_H^{(0)} = \hat{k}\times\nabla T^{(0)}, \qquad (3.7)$$

where $\mathbf{q}_H = \mathbf{q}-(\hat{k}\,.\,\mathbf{q})\,\hat{k}$. To this order, of course, $\mathbf{q}_H^{(0)} = \mathbf{q}^{(0)}$. The variation of the $O(1)$ (and similarly $O(E^{\frac{1}{2}})$) interior velocity in a direction parallel to the rotation axis is produced by horizontal density gradients. Thus, as is well known, vertical shears of the horizontal velocity are now allowed and the fundamental constraint of the Taylor–Proudman theorem is broken. As a result, the fluid no longer need behave as a set of columns. It is convenient to rewrite (3.2) as follows:

$$\mathbf{q}^{(0)} = \tfrac{1}{2}\hat{k}\times\nabla p^{(0)}. \qquad (3.8)$$

Since $\mathbf{q}^{(0)}$ as expressed in (3.8) satisfies (3.4) and (3.5) identically, (3.2)–(3.5) are degenerate and we must consider the $O(E)$ equations in order to derive an equation for $p^{(0)}$. Had we not assumed that $\epsilon < E$, we would have had to consider the departures from the geostrophic and hydrostatic balance due to the inertial and/or the diffusion terms. However, since $\epsilon < E$ the inertial terms can be disregarded altogether.

The $O(E)$ interior equations are:

$$2\hat{k}\times\mathbf{q}^{(2)} = -\nabla p^{(2)} + T^{(2)}\hat{k} + \nabla^2\mathbf{q}^{(0)}, \tag{3.9}$$

$$\sigma S\hat{k}\,.\,\mathbf{q}^{(2)} = \nabla^2 T^{(0)}, \tag{3.10}$$

$$\nabla\,.\,\mathbf{q}^{(2)} = 0. \tag{3.11}$$

The $O(E)$ vertical velocity depends for its existence on the diffusion of temperature in the fluid interior. The equation governing the $O(1)$ fields can be obtained by first forming the vertical component of the vorticity equation, viz.

$$2\frac{\partial w^{(2)}}{\partial z} + \nabla^2\zeta^{(0)} = 0, \tag{3.12}$$

where $w = \hat{k}\,.\,\mathbf{q}$ and $\zeta = \hat{k}\,.\,(\nabla\times\mathbf{q})$. Then using the heat equation (3.10) to eliminate $w^{(2)}$, we get

$$\nabla^2\left(\zeta^{(0)} + \frac{2}{\sigma S}T^{(0)}_z\right) = 0. \tag{3.13}$$

(3.12) states that the production of vorticity by vortex tube stretching in the direction of the rotation axis by means of the $O(E)$ vertical velocity must be balanced in the interior by the viscous dissipation of the $O(1)$ vorticity. Since $T^{(0)}$ and $\mathbf{q}^{(0)}$ can easily be deduced from $p^{(0)}$, it is preferable to rewrite (3.13) in terms of $p^{(0)}$, viz.

$$\nabla^2\left(\tfrac{1}{4}\sigma S\nabla_1^2 + \frac{\partial^2}{\partial z^2}\right)p^{(0)} = 0, \tag{3.13$'$}$$

where ∇_1^2 is the two-dimensional horizontal Laplacian operator. In the interior *the $O(1)$ motion is therefore completely controlled by the small diffusion present in the $O(1)$ motion.*

In order to formulate an 'interior' boundary-value problem, it is necessary to specify boundary conditions for the *interior* fields. This can only be done by a consideration of the various boundary-layer corrections to the interior fields which are needed to satisfy the boundary conditions (2.4)–(2.6).

4. The Ekman boundary layers

Viscous boundary layers exist in regions of thickness $E^{\frac{1}{2}}$ near the surfaces $z = 0$ and $z = 1$. In the region near $z = 0$, for example, we represent the dynamical variables in the following series:

$$\left.\begin{aligned}
\mathbf{q} &= \mathbf{q}^{(0)}(r,\theta,z) + E^{\frac{1}{2}}\mathbf{q}^{(1)}(r,\theta,z) + E\mathbf{q}^{(2)}(r,\theta,z) + \ldots \\
&\qquad + \tilde{\mathbf{q}}^{(0)}(r,\theta,\eta) + E^{\frac{1}{2}}\tilde{\mathbf{q}}^{(1)}(r,\theta,\eta) + \ldots, \\
p &= p^{(0)}(r,\theta,z) + E^{\frac{1}{2}}p^{(1)}(r,\theta,z) + Ep^{(2)}(r,\theta,z) + \ldots \\
&\qquad + E^{\frac{1}{2}}\tilde{p}^{(1)}(r,\theta,\eta) + \ldots, \\
T &= T^{(0)}(r,\theta,z) + E^{\frac{1}{2}}T^{(1)}(r,\theta,z) + ET^{(2)}(r,\theta,z) + \ldots \\
&\qquad + \tilde{T}^{(0)}(r,\theta,\eta) + E^{\frac{1}{2}}\tilde{T}^{(1)}(r,\theta,\eta) + \ldots,
\end{aligned}\right\} \tag{4.1}$$

where the tilde variables are boundary-layer corrections to the interior fields and are functions of the stretched variable

$$\eta = E^{-\frac{1}{2}}z.$$

Note that the boundary-layer correction to the pressure is of $O(E^{\frac{1}{2}})$. All the tilde variables must, of course, vanish as $\eta \to \infty$.

If (4.1) is substituted in (2.1)–(2.3), we obtain the boundary-layer equations for the $O(1)$ variables:

$$2\hat{k} \times \tilde{\mathbf{q}}_H^{(0)} = \frac{\partial^2}{\partial\eta^2}\tilde{\mathbf{q}}_H^{(0)}, \tag{4.2}$$

$$\tilde{T}^{(0)} = \frac{\partial\tilde{p}^{(1)}}{\partial\eta} - \frac{\partial^2\tilde{w}^{(0)}}{\partial\eta^2}, \tag{4.3}$$

$$\frac{\partial\tilde{w}^{(0)}}{\partial\eta} = 0, \tag{4.4}$$

$$\sigma S\tilde{w}^{(0)} = \frac{\partial^2}{\partial\eta^2}\tilde{T}^{(0)}. \tag{4.5}$$

From (4.3), (4.4) and (4.5) we immediately deduce that

$$\left.\begin{aligned}\tilde{w}^{(0)} &\equiv 0,\\ \tilde{T}^{(0)} &\equiv 0,\\ \tilde{p}^{(1)} &\equiv 0.\end{aligned}\right\} \tag{4.6}$$

Therefore, *the horizontal boundary layers are indeed the same Ekman layers as are present in a homogeneous fluid.* This is, of course, due to the fact that the Ekman-layer thickness is small compared to the stratification height. The solution of (4.2) subject to the conditions (2.4) is

$$\tilde{\mathbf{q}}^{(0)} = -(\mathbf{q}^{(0)} - \mathbf{q}_B)\,e^{-\eta}\cos\eta + \hat{k} \times (\mathbf{q}^{(0)} - \mathbf{q}_B)\,e^{-\eta}\sin\eta, \tag{4.7}$$

where the interior velocity $\mathbf{q}^{(0)}$ is evaluated at $z = 0$. An examination of the $O(E^{\frac{1}{2}})$ boundary-layer equations shows that

$$\partial\tilde{w}^{(1)}/\partial\eta = -\nabla\cdot\tilde{\mathbf{q}}^{(0)}, \tag{4.8}$$

or using (4.7),
$$\partial\tilde{w}^{(1)}/\partial\eta = (\zeta^{(0)} - \zeta_B)\,e^{-\eta}\sin\eta, \tag{4.9}$$

where
$$\zeta_B = \hat{k}\cdot\nabla\times\mathbf{q}_B.$$

Since $\tilde{w}^{(1)} + w^{(1)}$ is zero on $z = 0$, and since $w^{(1)}$ is identically zero because of the influence of the stratification on the interior vertical velocity, we deduce from (4.9) that

$$\tilde{w}^{(1)}(r,\theta,0) = \tfrac{1}{2}\{\zeta^{(0)}(r,\theta,0) - \zeta_B\} = 0,$$

or, expressing $\zeta^{(0)}$ in terms of $p^{(0)}$

$$\tfrac{1}{2}\nabla_1^2 p^{(0)} = \zeta_B \quad \text{on} \quad z = 0. \tag{4.10}$$

Thus, because of the stratification, the suction of fluid out of (or into) the Ekman boundary layers is zero to $O(E^{\frac{1}{2}})$. The stretching of fluid columns by an $O(E^{\frac{1}{2}})$ suction, which constitutes such an important mechanism in the theory of homogeneous rotating fluids, is absent. The Ekman layer plays, at most, a passive role in *the O(1) dynamics.* Note that if the prescribed velocity at $z = 0$ is irrotational ($\zeta_B = 0$), the interior velocity must also be irrotational on the plane $z = 0$, i.e.

$$\nabla_1^2 p^{(0)} = 0.$$

Using (4.10), (4.9) and (2.3) it can easily be shown that $T^{(1)}$ is zero, so that the interior temperature must satisfy the boundary condition (2.6*a*) by itself, i.e.

$$a_B \frac{\partial p^{(0)}}{\partial z} + b_B \frac{\partial^2 p^{(0)}}{\partial z^2} = \Theta_B(r,\theta) \quad \text{on} \quad z = 0. \tag{4.11}$$

The analysis of the Ekman layer on $z = 1$ proceeds in a similar manner, yielding the following boundary condition for the $O(1)$ interior flow:

$$\tfrac{1}{2}\nabla_1^2 p^{(0)} = \zeta_T \quad \text{on} \quad z = 1, \tag{4.12}$$

$$a_T \frac{\partial p^{(0)}}{\partial z} + b_T \frac{\partial^2 p^{(0)}}{\partial z^2} = \Theta_T(r,\theta) \quad \text{on} \quad z = 1. \tag{4.13}$$

In order to complete our formulation of the interior boundary-value problem, we now turn to a consideration of the vertical boundary layers.

5. The vertical boundary layers

In the theory of homogeneous, rotating fluids the side wall boundary layer has a double structure, and is made up of two layers of thicknesses $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ (Stewartson 1957). On account of the stratification neither of these layers exists in the present case. Instead, they are replaced by a single layer of thickness of $O(E^{\frac{1}{2}})$.

Consider the boundary layer in the region near $r = r_1$, i.e. the outer boundary. Let u, v and w be the radial, circumferential and vertical components of $\mathbf{q}$. In this side wall region we can represent the various fields as follows:

$$\left.\begin{aligned}
u &= u^{(0)}(r,\theta,z) + E^{\frac{1}{2}}u^{(1)}(r,\theta,z) + \ldots \\
&\qquad + E^{\frac{1}{2}}\bar{u}^{(1)}(\mu,\theta,z) + E\bar{u}^{(2)}(\mu,\theta,z) + \ldots, \\
v &= v^{(0)}(r,\theta,z) + E^{\frac{1}{2}}v^{(1)}(r,\theta,z) + \ldots \\
&\qquad + E^{\frac{1}{2}}\bar{v}^{(1)}(\mu,\theta,z) + E\bar{v}^{(2)}(\mu,\theta,z) + \ldots, \\
w &= Ew^{(2)}(r,\theta,z) + \ldots + \bar{w}^{(0)}(\mu,\theta,z) + E^{\frac{1}{2}}\bar{w}^{(1)}(\mu,\theta,z) + \ldots, \\
p &= p^{(0)}(r,\theta,z) + E^{\frac{1}{2}}p^{(1)}(r,\theta,z) + \ldots \\
&\qquad + E^{\frac{1}{2}}\bar{p}^{(1)}(\mu,\theta,z) + E\bar{p}^{(2)}(\mu,\theta,z) + \ldots, \\
T &= T^{(0)}(r,\theta,z) + E^{\frac{1}{2}}T^{(1)}(r,\theta,z) + \ldots \\
&\qquad + \bar{T}^{(0)}(\mu,\theta,z) + E^{\frac{1}{2}}\bar{T}^{(1)}(\mu,\theta,z) + \ldots .
\end{aligned}\right\} \tag{5.1}$$

The quantities with an over bar are the boundary-layer corrections which vanish exponentially as

$$\mu = E^{-\frac{1}{2}}(r_1 - r)$$

becomes infinite. Note that the corrections to the radial and circumferential velocities are at most of $O(E^{\frac{1}{2}})$.

Substitution of (5.1) into (2.1), (2.2) and (2.3) yields

$$0 = \bar{p}^{(1)}_{\mu}, \tag{5.2}$$

$$0 = -r_1^{-1}\bar{p}^{(1)}_{\theta} - 2\bar{u}^{(1)} + \bar{v}^{(1)}_{\mu\mu}, \tag{5.3}$$

$$0 = \bar{T}^{(0)} + \bar{w}^{(0)}_{\mu\mu}, \tag{5.4}$$

$$0 = \sigma S \bar{w}^{(0)} - \bar{T}^{(0)}_{\mu\mu}, \tag{5.5}$$

$$-\bar{u}^{(1)}_{\mu} + \bar{w}^{(0)}_{z} = 0. \tag{5.6}$$

Since $\bar{p}^{(1)}$ vanishes at infinity, (5.2) implies that $\bar{p}^{(1)}$ is identically zero. Thus, once $\bar{w}^{(0)}$ and $\bar{T}^{(0)}$ are determined, $\bar{u}^{(1)}$ and $\bar{v}^{(1)}$ can be found by means of (5.6) and (5.3). The primary variables in this layer are $\bar{w}^{(0)}$ and $\bar{T}^{(0)}$, and it is interesting to note that the boundary-layer equations in these variables have the same structure as the Ekman-layer equations for $\tilde{u}^{(0)}$ and $\tilde{v}^{(0)}$ in the neighbourhood of $z = 0, 1$. Because $w^{(0)}$ is identically zero, $\bar{w}^{(0)}$ must vanish on $\mu = 0$; consequently, we deduce that

$$\bar{w}^{(0)} = C(\theta, z)\, e^{-k\mu} \sin k\mu, \tag{5.7}$$

$$\bar{T}^{(0)} = 2k^2 C(\theta, z)\, e^{-k\mu} \cos k\mu, \tag{5.8}$$

$$\bar{u}^{(1)} = -\frac{1}{k\sqrt{2}}(\partial C/\partial z)\, e^{-k\mu} \sin(k\mu + \tfrac{1}{4}\pi), \tag{5.9}$$

$$\bar{v}^{(1)} = \frac{1}{k^3\sqrt{2}}(\partial C/\partial z)\, e^{-k\mu} \cos(k\mu + \tfrac{1}{4}\pi), \tag{5.10}$$

where $k = (\sigma S)^{\frac{1}{4}}/\sqrt{2}$, and $C(\theta, z)$ is yet to be determined.

It is clear that the interior radial and circumferential velocities must satisfy the boundary conditions (2.5) by themselves, viz.

$$\partial p^{(0)}/\partial\theta = 0 \quad \text{on} \quad r = r_1, \tag{5.11}$$

$$\partial p^{(0)}/\partial r = 2V_o \quad \text{on} \quad r = r_1. \tag{5.12}$$

The thermal boundary condition (2.6*c*) may now be written thus:

(i) If $b_o \neq 0$,

$$\left.\begin{aligned} \partial\bar{T}^{(0)}/\partial\mu &= 0, \\ a_o\bar{T}^{(0)} - b_o\,\partial\bar{T}^{(1)}/\partial\mu &= \Theta_o - a_o\,\partial p^{(0)}/\partial z - b_o\,\partial^2 p^{(0)}/\partial z\,\partial r. \end{aligned}\right\} \tag{5.13}$$

(ii) If $b_o = 0$,

$$a_o\bar{T}^{(0)} = \Theta_o - a_o\,\partial p^{(0)}/\partial z. \tag{5.13'}$$

If the region is not simply-connected (i.e. $r_2 \neq 0$), a similar boundary layer exists on the inner wall, yielding the following boundary-layer corrections:

$$\bar{w}^{(0)} = D(\theta, z)\, e^{-k\lambda} \sin k\lambda, \tag{5.14}$$

$$\bar{T}^{(0)} = 2k^2 D(\theta, z)\, e^{-k\lambda} \cos k\lambda, \tag{5.15}$$

$$\bar{u}^{(1)} = \frac{1}{k\sqrt{2}}(\partial D/\partial z)\, e^{-k\lambda} \sin(k\lambda + \tfrac{1}{4}\pi), \tag{5.16}$$

$$\bar{v}^{(1)} = \frac{-1}{k^3\sqrt{2}}(\partial D/\partial z)\, e^{-k\lambda} \cos(k\lambda + \tfrac{1}{4}\pi), \tag{5.17}$$

where $$\lambda = E^{-\frac{1}{2}}(r - r_2).$$

The boundary conditions on $r = r_2$ are analogous to (5.11)–(5.13).

Further discussion now depends on the particular type of thermal boundary conditions.

6. Conditions on the interior flow

6.1. *Simply connected region*

Let us first consider the simpler case in which the region is simply connected, i.e. $r_2 = 0$. Then the only vertical boundary layer occurs on $r = r_1$.

We have already seen that on $z = 0$ and $z = 1$, the $O(1)$ pressure must satisfy the relations:

$$\tfrac{1}{2}\nabla_1^2 p^{(0)} = \zeta_B \quad \text{on} \quad z = 0, \tag{6.1 a}$$

$$\tfrac{1}{2}\nabla_1^2 p^{(0)} = \zeta_T \quad \text{on} \quad z = 1. \tag{6.1 b}$$

In addition, since the $O(1)$ pressure is the interior pressure, $p^{(0)}$ must be continuous at the rims of the container. This, together with (5.11), implies that on $z = 0$ and $z = 1$, the above two-dimensional Poisson equations must be solved subject to the condition that $p^{(0)}$ be a constant on the circular boundary of the planar region. Since the regions are simply connected, the only solutions of (6.1), written in terms of the velocity, are

$$\left.\begin{aligned} \mathbf{q}^{(0)}(r,\theta,0) &= \mathbf{q}_B(r,\theta), \\ \mathbf{q}^{(0)}(r,\theta,1) &= \mathbf{q}_T(r,\theta). \end{aligned}\right\} \tag{6.2}$$

Thus in a *simply connected region* the *interior* $O(1)$ velocities must match the prescribed boundary velocity. *To this order there is no Ekman layer*; this follows formally from (6.2) and (4.7). It is important to note that the interior problem is of sufficiently high order to be able to meet this condition. Thus, for a simply connected region, there is generally no Taylor–Proudman theorem, no Ekman layer suction and no Ekman-layer.

Let us turn our attention to the side wall boundary layer. We shall consider separately the two cases where first, a condition is solely placed on the temperature, or density ($b_o = 0$), and secondly, where a condition is placed on the heat flux ($b_o \neq 0$).

Putting b_o equal to zero, and using (5.8), the thermal boundary condition (5.13) can be written thus

$$2k^2 C(\theta, z)\, a_o = \Theta_o(\theta, z) - a_o \frac{\partial p^{(0)}}{\partial z} \quad \text{on} \quad r = r_1. \tag{6.3}$$

It is convenient to write both $C(\theta, z)$ and $\Theta(\theta, z)/a_o$ as Fourier series, viz.

$$C(\theta, z) = C_0(z) + \sum_{n=1}^{\infty} \{C_n^c(z)\cos n\theta + C_n^s(z)\sin n\theta\}, \tag{6.4}$$

$$\Theta_o(\theta, z)/a_o = \mathscr{T}_0(z) + \sum_{n=1}^{\infty} \{\mathscr{T}_n^c(z)\cos n\theta + \mathscr{T}_n^s(z)\sin n\theta\}. \tag{6.5}$$

Using the continuity equation (5.6) and the fact that $u^{(1)}$ is geostrophic, we can deduce that

$$\frac{\partial}{\partial z}\int_0^{2\pi} d\theta \int_0^{\infty} \overline{w}^{(0)}\, d\mu = 0, \tag{6.6}$$

which simply states that the net vertical $E^{\frac{1}{2}}$ transport in the side wall boundary layer, across any horizontal plane is constant. Upon using (5.7) and (6.4), (6.6) implies that

$$\partial C_0/\partial z = 0, \tag{6.7}$$

i.e. C_0 is a constant. In a simply connected region this constant must obviously be zero, but this need not be the case for a multiply connected region. Averaging now (6.3) over θ, and using the condition (5.11), we deduce that

$$\partial p^{(0)}/\partial z = \mathscr{T}_0(z), \tag{6.8}$$

which completes the specification of the interior boundary-value problem. In addition, we can also derive from (6.3) that

$$2k^2 C_n^c(z) = \mathscr{T}_n^c(z), \tag{6.9a}$$

$$2k^2 C_n^s(z) = \mathscr{T}_n^s(z). \tag{6.9b}$$

Any lack of axial symmetry in the prescribed temperature is taken up completely by the side-wall boundary layer, and *the interior temperature (density) satisfies an axially symmetric condition on* $r = r_1$, viz. (6.8). The asymmetry in the prescribed temperature affects the fluid interior only in so far as it produces a flux of $O(E^{\frac{1}{2}})$ in the side-wall layer which must penetrate the interior. Thus, a cylinder heated on one side and cooled on the other would produce rising currents on the hot side, descending currents on the cool side and the circulation will be closed by an $O(E^{\frac{1}{2}})$ interior motion. Finally, it should be noted that, although $C_0 \equiv 0$ insures that there is no θ-average flux in or out of the vertical layer, there exists the possibility that $\overline{w}^{(0)} \neq 0$ at $z = 0, 1$. If this were indeed the case it would imply the existence of an $O(E^{-\frac{1}{2}})$ azimuthal velocity within the narrow 'corner' regions (since there can be no $O(E^{\frac{1}{2}})$ flux in or out of the Ekman layers). However, this $O(E^{-\frac{1}{2}})$ corner-region correction to the azimuthal velocity which satisfies a homogeneous set of equations with homogeneous boundary conditions, is identically zero and consequently we must require that $\overline{w}^{(0)} = 0$ at $z = 0, 1$. This requirement can be satisfied only if the prescribed temperature $\Theta_0(\theta, z)$ is a constant for $z = 0, 1$. More general heatings are outside the scope of this linear theory, since they would imply the existence of regions in which the inertial forces cannot be neglected.

The case in which the thermal boundary condition along the vertical wall involves the heat flux (i.e. $b_o \neq 0$) can be treated as follows. From the first equation in (5.13) and (5.8) it is clear that $C(\theta, z)$ is identically zero. Therefore the first non-zero boundary-layer corrections are of lower order than for the case $b_o = 0$, but still of the same form, viz.

$$\overline{w}^{(1)} = C^{(1)}(0, z)\, e^{-k\mu} \sin k\mu,$$

$$\overline{T}^{(1)} = 2k^2 C^{(1)}(0, z)\, e^{-k\mu} \cos k\mu,$$

$$\overline{u}^{(2)} = -\frac{1}{k\sqrt{2}} \frac{\partial C^{(1)}}{\partial z} e^{-k\mu} \sin (k\mu + \tfrac{1}{4}\pi),$$

$$\overline{v}^{(2)} = \frac{1}{k^3\sqrt{2}} \frac{\partial C^{(1)}}{\partial z} e^{-k\mu} \cos (k\mu + \tfrac{1}{4}\pi).$$

Using the above expression for $\overline{T}^{(1)}$ together with (5.12), the thermal boundary condition (5.13) can now be written thus

$$2k^3 b_o C^{(1)} = \Theta_o - a_o \frac{\partial p^{(0)}}{\partial z} - 2b_o \frac{\partial V_o}{\partial z}.$$

Or, averaging this expression and recalling that $p^{(0)}$ is not a function of θ on the boundary, we get

$$2k^3 b_o \langle C^{(1)}\rangle = \langle \Theta_o\rangle - a_o \, \partial p^{(0)}/\partial z - 2b_o \, \partial \langle V_o\rangle/\partial z \quad \text{on} \quad r = r_1. \tag{6.3'}$$

An additional relation between $\langle C^{(1)}\rangle$ and $p^{(0)}$ can be obtained from the $O(E)$ boundary condition for the radial velocity, or rather from its θ-average, viz.

$$\langle u^{(2)}\rangle = -\langle \overline{u}^{(2)}\rangle \quad \text{on} \quad r = r_1.$$

Using the expression for $u^{(2)}$ given by the azimuthal component of the momentum equation and the expression for $\overline{u}^{(2)}$ in terms of $C^{(1)}$, we get

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p^{(0)}}{\partial r} + 2\frac{\partial^2}{\partial r}\langle V_o\rangle = \frac{2}{k}\frac{\partial}{\partial z}\langle C^{(1)}\rangle.$$

Eliminating $\langle C^{(1)}\rangle$ between the above equation and (6.3′) we deduce that

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p^{(0)}}{\partial r} = k^{-4}\frac{\partial}{\partial z}\left\{ b_o^{-1}\left(\langle \Theta_o\rangle - a_o \frac{\partial p^{(0)}}{\partial z}\right) - 2(1+k^4)\frac{\partial}{\partial z}\langle V_o\rangle\right\}, \tag{6.8'}$$

which together with (5.12) constitute the appropriate boundary conditions for $p^{(0)}$. Note that if the wall is insulated and fixed in the rotating frame, i.e. if $\Theta_0 = a_o = V_0 = 0$, this last boundary condition becomes

$$\frac{\partial}{\partial r} r \frac{\partial^2 p^{(0)}}{\partial r^2} = 0. \tag{6.8''}$$

The situation regarding the restrictions on Θ_o differs slightly from the one above, valid for the case $b_o = 0$. In particular, if the solution of the $O(1)$ interior boundary-value problem is not identically zero (i.e. if the horizontal boundary conditions and/or the dynamical vertical boundary conditions are inhomogeneous), it is not necessary to restrict the form of Θ_o. This is due to the fact that the radial velocity $\tilde{u}^{(1)}$ in the Ekman layers can accommodate a θ-dependent flux. However, if the motion is solely due to Θ_o (i.e. if the interior $O(1)$ temperature is identically zero) then by means of arguments analogous to those used above, one can show that $\overline{w}^{(1)}$ must vanish at $z = 0, 1$ and therefore that Θ_o must be a constant along the rims of the container.†

6.2. *Doubly connected region*

Let us now turn our attention to the more complicated case in which the region is doubly connected, i.e. $r_2 \neq 0$. We shall see that the annular geometry gives the fluid motion an added degree of flexibility.

† Both in this case and in the analogous one for $b_o = 0$, the representation inside the vertical boundary layer is valid throughout the entire height of the container and, since there are no Ekman layers (of comparable order), 'corner regions' are absent.

Let us rewrite the general conditions for $p^{(0)}$ at $z = 0$ and $z = 1$, which stem from the suppression of the vertical velocity, viz.

$$\tfrac{1}{2}\nabla_1^2 p^{(0)} = \zeta_B \quad \text{on} \quad z = 0, \tag{6.10a}$$

$$\tfrac{1}{2}\nabla_1^2 p^{(0)} = \zeta_T \quad \text{on} \quad z = 1. \tag{6.10b}$$

From (5.11) and (5.18), it follows that we must solve the Poisson equations (6.10) for $p^{(0)}$ subject to the conditions that

$$\partial p^{(0)}/\partial\theta = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2. \tag{6.11}$$

However, because the horizontal boundaries are doubly connected, (6.10) and (6.11) *do not specify a unique solution*. To define the solution uniquely it is necessary to specify the circulation of the interior flow in the planes $z = 0$ and $z = 1$. If we define the functions $\psi_B(r,\theta)$ and $\psi_T(r,\theta)$ such that

$$\mathbf{q}_B = \hat{k}\times\nabla\psi_B, \tag{6.12a}$$

$$\mathbf{q}_T = \hat{k}\times\nabla\psi_T, \tag{6.12b}$$

then, the most general solution of (6.10) subject to (6.11) is

$$p^{(0)} = 2\psi_B + 2A_B \ln r + K_B \quad \text{on} \quad z = 0; \tag{6.13a}$$

$$p^{(0)} = 2\psi_T + 2A_T \ln r + K_T \quad \text{on} \quad z = 1; \tag{6.13b}$$

where A_B and A_T are measures of the interior circulations on the planes $z = 0$ and $z = 1$. The corresponding velocities are:

$$\mathbf{q}^{(0)} = \mathbf{q}_B + \frac{A_B}{r}\hat{\theta} \quad \text{on} \quad z = 0; \tag{6.14a}$$

$$\mathbf{q}^{(0)} = \mathbf{q}_T + \frac{A_T}{r}\hat{\theta} \quad \text{on} \quad z = 1. \tag{6.14b}$$

For a simply connected region A_B and A_T must be zero in order to prevent the velocity from being infinite at $r = 0$. In a multiply connected region this need not be the case, and it raises the possibility that the interior velocity does not match the boundary conditions, and consequently that $O(1)$ Ekman layers exist. Of course, even in this case there is no Ekman-layer suction.

In order to determine whether A_T and A_B are non-zero let us first consider the case in which the thermal condition on at least one side wall involves the heat flux. For definiteness, let $b_o \neq 0$. Then, from (5.13) we deduce that

$$C(\theta, z) = 0. \tag{6.15}$$

This means that the net vertical flux in the outer side wall layers is at most of $O(E)$. On the other hand the radial flux in the lower Ekman layer

$$\int_0^{2\pi} r\,d\theta \int_0^{\infty} \tilde{u}\,d\eta$$

is, using (4.7) and (6.14*a*), $\quad -\pi A_B E^{\frac{1}{2}}.$

From flux continuity we must therefore require that $A_B = 0$, and similarly that $A_T = 0$. Thus, even in this multiply connected region where an Ekman layer is *a priori* possible, a detailed consideration of the side wall layer shows that, in fact, no Ekman layer exists if the thermal condition on *either* side wall involves the heat flux. The interior variables must therefore again satisfy all the prescribed conditions on $z = 0$ and $z = 1$, while the interior velocity must match the side velocity conditions.

The final case to consider occurs when the region is doubly connected and *neither* inner nor outer wall has a thermal boundary condition involving the heat flux. In this case $b_o = b_I = 0$. Then, on the outer wall

$$2k^2C(\theta, z) = \frac{\Theta_o(\theta, z)}{a_o} - \frac{\partial p^{(0)}}{\partial z} \quad (r = r_1), \tag{6.16a}$$

while on the inner

$$2k^2D(\theta, z) = \frac{\Theta_I(\theta, z)}{a_I} - \frac{\partial p^{(0)}}{\partial z} \quad (r = r_2). \tag{6.16b}$$

As before, we write

$$C(\theta, z) = C_0 + \sum_{n=1}^{\infty} \{C_n^c(z)\cos n\theta + C_n^s(z)\sin n\theta\}, \tag{6.17a}$$

$$\frac{\Theta_o(\theta, z)}{a_o} = \mathscr{T}_{o0}(z) + \sum_{n=1}^{\infty} \{\mathscr{T}_{on}^c(z)\cos n\theta + \mathscr{T}_{on}^s(z)\sin n\theta\}, \tag{6.18a}$$

and similarly

$$D(\theta, z) = D_0 + \sum_{n=1}^{\infty} \{D_n^c(z)\cos n\theta + D_n^s(z)\sin n\theta\}, \tag{6.17b}$$

$$\frac{\Theta_I(\theta, z)}{a_I} = \mathscr{T}_{I0}(z) + \sum_{n=1}^{\infty} \{\mathscr{T}_{In}^c(z)\cos n\theta + \mathscr{T}_{In}^s(z)\sin n\theta\}. \tag{6.18b}$$

As previously, see (6.7), *C_0 and D_0 are independent of z* since the vertical boundary layers are, on the mean, non-divergent. Recalling that $p^{(0)}$ is independent of θ both on $r = r_1$ and $r = r_2$, we can now deduce from (6.16) that

$$C_n^c = (2k^2a_o)^{-1}\mathscr{T}_{on}^c, \quad C_n^s = (2k^2a_o)^{-1}\mathscr{T}_{on}^s, \tag{6.19a, b}$$

$$D_n^c = (2k^2a_I)^{-1}\mathscr{T}_{In}^c, \quad D_n^s = (2k^2a_I)^{-1}\mathscr{T}_{In}^s \tag{6.19c, d}$$

and

$$\frac{\partial p^{(0)}}{\partial z} = \mathscr{T}_{o0}(z) - 2k^2C_0 \quad \text{on} \quad r = r_1, \tag{6.20a}$$

$$\frac{\partial p^{(0)}}{\partial z} = \mathscr{T}_{I0}(z) - 2k^2D_0 \quad \text{on} \quad r = r_2. \tag{6.20b}$$

Once again, *the interior $O(1)$ temperature 'sees' only the axially symmetric portion of the prescribed side wall temperature*, the remaining going into the production of vertical side wall layer velocities.

(6.20) may be integrated to yield the pressure on the side walls, viz.

$$p^{(0)}(r_1, \theta, z) = \int_0^z \mathscr{T}_{o0}(z')\,dz' - 2k^2C_0z + K_o, \tag{6.21a}$$

and

$$p^{(0)}(r_2, \theta, z) = \int_0^z \mathscr{T}_{I0}(z')\,dz' - 2k^2D_0z + K_I, \tag{6.21b}$$

while the pressure on $z = 0$ and $z = 1$ is given by (6.13). In order to complete the formulation of the interior boundary-value problem, we must evaluate A_T, A_B,

C_0, D_0 and the K's. The $O(E^{\frac{1}{2}})$ net mass fluxes in the outer and inner side wall layers are

$$F_o = \frac{C_0}{2k} 2\pi r_1 E^{\frac{1}{2}}, \quad F_I = \frac{D_0}{2k} 2\pi r_2 E^{\frac{1}{2}}, \tag{6.22a, b}$$

while the radial fluxes in the top and bottom Ekman layers are

$$F_T = -\pi A_T E^{\frac{1}{2}}, \quad F_B = -\pi A_B E^{\frac{1}{2}}. \tag{6.23a, b}$$

From flux continuity we must require that

$$r_1 k^{-1} C_0 = -A_B, \tag{6.24}$$

$$r_2 k^{-1} D_0 = -A_T, \tag{6.25}$$

$$A_T = -A_B, \tag{6.26}$$

from which we deduce that

$$C_0 = -D_0 r_2/r_1 = -kA_B/r_1. \tag{6.27}$$

An additional equation for the four flux measures C_0, D_0, A_T and A_B is obtained from pressure-continuity requirements. Indeed, at the rims of the annular region we must have

$$p^{(0)}(r_1, \theta, 1) = \int_0^1 \mathscr{T}_{o0}(z')\,dz' - 2k^2 C_0 + K_0 = 2\psi_T(r_1) + 2A_T \ln r_1 + K_T, \tag{6.28a}$$

$$p^{(0)}(r_1, \theta, 0) = K_o = 2\psi_B(r_1) + 2A_B \ln r_1 + K_B, \tag{6.28b}$$

$$p^{(0)}(r_2, \theta, 0) = K_I = 2\psi_B(r_2) + 2A_B \ln r_2 + K_B, \tag{6.28c}$$

$$p^{(0)}(r_2, \theta, 1) = \int_0^1 \mathscr{T}_{I0}(z')\,dz' - 2k^2 D_0 + K_I = 2\psi_T(r_2) + 2A_T \ln r_2 + K_T. \tag{6.28d}$$

Eliminating the K's in (6.28) and using (6.27), we obtain

$$A_B = \frac{\frac{1}{2}[\{\psi_T(r_1) - \psi_T(r_2)\} - \{\psi_B(r_1) - \psi_B(r_2)\}] - \frac{1}{4}\int_0^1 \{\mathscr{T}_{o0}(z') - \mathscr{T}_{I0}(z')\}\,dz'}{\ln\frac{r_1}{r_2} + \frac{k^3}{2}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\}}. \tag{6.29}$$

A special case of interest occurs when the boundaries are fixed. In this case (6.29) reduces to

$$A_B = -A_T = -\frac{1}{4}\left[\int_0^1 \{\mathscr{T}_{o0}(z') - \mathscr{T}_{I0}(z')\}\,dz'\right] \Big/ \left[\ln\frac{r_1}{r_2} + \frac{k^3}{2}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\}\right]. \tag{6.30}$$

Since $\mathscr{T}_{o0}$ and $\mathscr{T}_{I0}$ are related to the axially symmetric part of the imposed side wall temperature conditions, (6.30) states that there will be a circumferential, interior circulation at the horizontal boundaries when the average of the temperatures of the inner and outer walls differ. Otherwise $A_B = A_T = 0$, and there will be no net transport in the Ekman layer and, in fact, no Ekman layers. It is interesting to note that the existence of the cited circumferential circulation (and Ekman layers) depends on the *average* of the imposed temperature differences. The relation (6.30) can also be considered as a net overall thermal wind balance for the region; such a consideration rationalizes the importance of the difference of the average of the imposed temperature differences. Barcilon (1962), in an earlier investigation, found results which are consistent with this analysis in his detailed consideration of the heated annulus problem.

In conclusion, we find that the stratification introduces important modifications in the dynamics of rotating fluids. In particular, except in the case just referred to of an annular region with no heat flux constraints on the side walls, all stratified, low Rossby number, rotating fluids *have no Ekman layers*. Even in the above-mentioned case, certain conditions must be met if there are to be $O(1)$ Ekman layers, namely (6.29). The interior flow is diffusive and, although geostrophic, the presence of interior dissipation leads to a mathematical problem of sufficiently high order to satisfy all the boundary conditions when, as is usually the case, the viscous boundary layers are absent. Finally, we should mention that the general results of the present paper, which is restricted to steady motions, are also valid for time-dependent flows, provided that the time scale of the motion is of $O(E^{-1}\Omega^{-1})$, i.e. a diffusive time scale.

7. Examples

We include here two very simple examples which illustrate the fundamental notions of the earlier sections and further display the basic differences between the homogeneous and stratified fluids. Further examples will be discussed in a later paper.

Example 1. Consider infinite horizontal disks rotating differentially. Let the disk at $z = 0$ be fixed in a system rotating with angular speed Ω, while the upper disk at $z = 1$ rotates with speed $\Omega(1+\epsilon)$. The well-known solution for the homogeneous fluid consists of an Ekman layer on each plate merging into an interior flow which is a circumferential solid body rotation with angular velocity $\Omega(1+\frac{1}{2}\epsilon)$, i.e. the mean of the two disks. The solution for a stratified fluid is easy to find. We consider here the case where the upper and lower plates are insulated, i.e.

$$\partial^2 p^{(0)}/\partial z^2 = 0 \quad \text{on} \quad z = 0, 1.$$

Then it can be verified that the expression

$$p^{(0)} = r^2 z,$$

which leads to

$$v^{(0)} = rz,$$

constitutes a solution† of the interior equation (3.13′) and of the imposed boundary conditions on the temperature and velocity at $z = 0, 1$. The circumferential velocity is no longer independent of z in the interior but changes linearly from the velocity of the upper to that of the lower plate. Furthermore, Ekman layers are absent.‡ The fluid interior no longer acts as a column but as a series of

† Unless the z-variation of the temperature is specified, the solution to the above boundary-value problem is not unique.

‡ This result differs from that of Carrier (1965) who considered a similar example. He finds that Ekman layers are present along horizontal boundaries. However, his analysis differs in so far as the assumptions on the Froude and Prandtl numbers are concerned.

'fluid disks', each slipping over its neighbour and transmitting the velocity applied by the differentially rotating disks by viscous stresses rather than vortex tube stretching due to Ekman-layer suction.

Example 2. If the region is again contained between two rotating disks, which are, however, finite, the problem is slightly more complex. Let the region be bounded at $r = r_1$ by a circular cylinder which is non-conducting. Then, our analysis implies that there is neither Ekman layer nor vertical side wall layer. The solution of (3.13′), subject to the conditions

$$\partial p^{(0)}/\partial r = \partial^2 p^{(0)}/\partial z^2 = 0 \quad \text{on} \quad z = 0,$$

$$\tfrac{1}{2}\,\partial p^{(0)}/\partial r - r = \partial^2 p^{(0)}/\partial z^2 = 0 \quad \text{on} \quad z = 1,$$

$$\partial p^{(0)}/\partial r = \partial[r\partial^2 p^{(0)}/\partial r^2]/\partial r = 0 \quad \text{on} \quad r = r_1,$$

is
$$p^{(0)} = \sum_{n=1}^{\infty} A_n \left\{\frac{\sinh k_n/r_1 z}{\sinh k_n/r_1} - \frac{\sinh k_n l/r_1 z}{l^2 \sinh k_n l/r_1}\right\} \frac{J_0[k_n(r/r_1)]}{1-l^{-2}} + \text{const.},$$

where
$$l = 2(\sigma S)^{-\frac{1}{2}}, \quad J_0'(k_n) = 0,$$

and
$$A_n = \frac{4r_1^2}{k_n^2}\Big/ J_0(k_n).$$

In the limit $r_1 \to \infty$, this solution approaches the solution of example 1. The fluid again behaves in a strikingly different manner than the homogeneous analogue.

This work was supported by the National Science Foundation under Contract No. GP 4321 and was also partially supported by the Office of Scientific Research of the U.S. Air Force, Grant AF-AFOSR-492-66.

REFERENCES

Barcilon, V. 1962 Thermally driven motion of a stably stratified fluid in a rotating annulus. Ph.D. thesis, Harvard University, 69 pp.

Carrier, G. F. 1965 Some effects on stratification and geometry in rotating fluids. *J. Fluid Mech.* **23**, 145–172.

Greenspan, H. P. 1964 On the transient motion of a contained rotating fluid. *J. Fluid Mech.* **20**, 673–696.

Greenspan, H. P. 1965 On the general theory of contained rotating fluid motions. *J. Fluid Mech.* **22**, 449–462.

Greenspan, H. P. & Howard, L. N. 1963 On a time dependent motion of a rotating fluid. *J. Fluid Mech.* **17**, 385–404.

Stewartson, K. 1957 On almost rigid rotation. *J. Fluid Mech.* **3**, 17–26.

Taylor, G. I. 1917 Motion of solids in fluids when the flow is irrotational. *Proc. Roy. Soc.* A **93**, 99–113.